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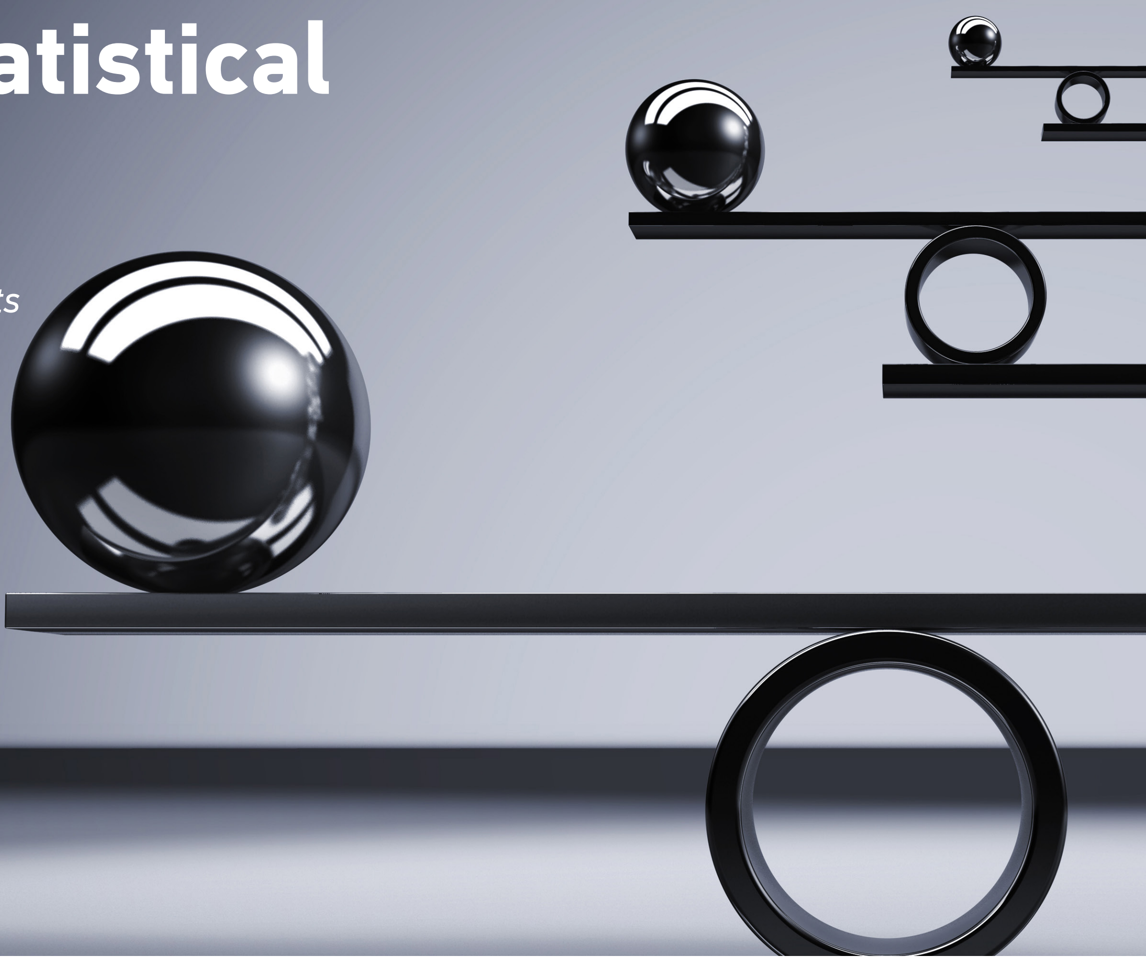
WORLDQUANT PERSPECTIVES

Zero-Sum and Other Statistical Games

Game theory offers insights into the strategies and decisions of human contestants engaged in competitive situations.



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THE EMERGENCE OF HUMANS IN THE LONG STORY OF EVOLUTION brought the world something novel: The growth of the human brain and the development of self-conscious thought led to the ability to think and to plan. The human animal operated not just by instinct but could imagine various futures, with multiple possible outcomes. For example, hunting in the age before agriculture was about survival and providing essential food for a nomadic tribe. It required organization and disciplined coordination to track and kill large animals. And that need for cooperation raised a number of salient issues that still confront us.

In *A Discourse on Inequality,* 18th-century French philosopher Jean-Jacques Rousseau famously sketched out what is now called the stag-hunt problem. In Rousseau's tale, a group of hunters go out in search of food and are immediately confronted by a challenge. If they want to maximize their hunt, they can wait for a large deer to trot by, and kill it. But that might involve hours of waiting, hidden in the bushes, with no certainty that a stag will appear. Meanwhile, rabbits are happily bouncing through the underbrush. Any of the hunters could leave the group and its hiding place to bag a rabbit, but that might scare off a shy stag. In short, an individual hunter could provide food for himself but the rest of the group would go hungry.

Today, in the field of game theory, Rousseau's tale has been modeled and explored for what it reveals about individuals and social cooperation. Hunters have a free choice, not informed by what anyone else is thinking. The odds of bagging a rabbit are greater for individuals but potentially destructive of the group. (British philosopher David Hume, a Rousseau contemporary, reimagined this situation as two people in a small boat that both must row to move ahead; if one stops, there's no reason for the other to continue. Hume used the story to explain how people learn what he called the convention of cooperation.) At the heart of the stag hunt is an elemental decision that everyone must make: Should the individual hunter give in to hunger or impatience, break ranks and satisfy his own needs at the cost of everyone else's? The decision to opt for the easy rabbit is what's known as a zero-sum game, in which one person's gain is another's loss.

A SHORT HISTORY OF GAME THEORY

Game theory is a mathematical field with diverse applications. It can be used to decide the best approach in competitions of various kinds. When trying to apply it to real-world situations, special attention should be paid to the nature of the game; a zero-sum game, for instance, requires fixed resources. Game theory, combined with computer science, provides a systematic, quantitative approach to choosing the best strategy in competitive situations, and it has become a powerful decision-making tool in many contexts, including investing and even management. However, game theory also has limitations, especially if the competition has many contestants. Game theory provides general rules for rational decision-making, but it doesn't necessarily lead to winning strategies.

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Consider a situation from the Babylonian Talmud, a collection of ancient Jewish laws and traditions compiled around 500 A.D. The Talmud wrestled with a marriage contract problem that was, in the end, a zero-sum game. In the scenario, a man has three wives whose marriage contracts state that in the case of his death they are entitled to receive 100, 200 and 300 units of his estate, respectively, depending on spousal longevity. But what if the man lacks funds to satisfy the contract? The rabbis who wrote the Talmud offered a series of recommendations for apportioning the estate. If the man dies leaving only 100 units, the Talmud recommends splitting them equally among the three wives; if he leaves 200, the wives should receive 50, 75 and 75; if he has 300, the estate should be split proportionally along the lines of the original example, with the wives receiving 50, 100 and 150. The rabbis were attempting to deal with the inequities of an estate with finite resources, and to ease some of the sting from an inevitably zero-sum game.

The first systematic studies of this type of decision-making came in 1921 when Émile Borel, a French mathematician and politician, tackled game theory. Borel correctly intuited that game theory could be used in economic and military applications, and he sought to discover the best strategy for a given situation. However, he failed to develop his ideas very far and his contribution to game theory has been overshadowed by others.

John von Neumann was a giant in his field. In 1928, as a postdoctoral student in Germany, the Hungarian-born mathematician and polymath published "On the Theory of Parlor Games." The article discussed game theory and gave a proof for what would become known as the minimax theorem. This involves a zero-sum game between two opponents: Player A knows that if she maximizes her return from the game, she must minimize that of her opponent, player B. The outcomes can be mathematically graphed, ranging from total wins by A or B to a situation where A and B find an equilibrium point, from which they can't rationally deviate without harming their own returns. During World War II, at the Institute for Advanced Study, in Princeton, New Jersey, von Neumann teamed up with Oskar Morgenstern, an Austrian economist, to develop game theory as it applied to economics. In 1944, they published *Theory of Games and Economic Behavior*, which, though written

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for economists, had applications in psychology, sociology, politics, warfare, recreational games and many other fields.

Von Neumann and Morgenstern's book prepared the ground for another Princeton mathematician, John Nash, to explore what's known as the Nash equilibrium. Nash was dealing with a zero-sum game, but with added complexities: Both players have their own strategy and know the strategy of their opponent. An equilibrium is reached when neither player can benefit from changing his strategy if his opponent's scheme remains unchanged; thus, you can predict the outcome of a zero-sum game by understanding the strategy of just one player. (The stag hunt has been modeled as a game with two Nash equilibria, depending on whether the strategies are risk dominant or payoff dominant.) For this achievement, John Nash shared the Nobel Memorial Prize in Economic Sciences in 1994. (Nash also suffered, then recovered, from paranoid schizophrenia, captured in the biography and popular movie *A Beautiful Mind.*)

THE COMPLEXITIES OF ZERO-SUM GAMES

Because a zero-sum game involves one person's gain and another's loss, the net change or overall benefit to the group is zero.[1,2] This type of game can have any number of players as long as it has at least two. To be a true zero-sum game, the total value of losses has to exactly equal the total value of gains. For zero-sum games, we assume that none of the players knows the strategy of the other players and that their decisions are rational.

Baccarat, blackjack, poker and roulette are popular zero-sum games.[3,4] Investing in stocks is a zero-sum game because the outperformance of one investment strategy requires the underperformance of other strategies.[5] There will always be a buyer and a seller of a stock, and the gain of one will involve a loss for the other. And investors' collective performance in the stock market relative to an index over a fixed period of time (usually a day) can also be viewed as a zero-sum game. Because the value of an index includes all gains and losses, it is by definition zero sum. However, the entire stock market should not be thought of as a zero-sum game because it does not meet the criterion of being a game with contestants. Why? Think about a company that issues shares on the market. An investor — let's call him Peter — buys some of those shares. But while the company raises equity from the shares, it does not usually participate in trading, and the market's resources are not fixed or well defined. Every day, there is a possibility that a new company will join the market by creating new shares or that a current company will go bankrupt. The game involves only an individual's performance relative to the stock market index.

People often inaccurately use the term "zero-sum game" to describe nongame situations and endeavors. For example, elections are not zero-sum games because the pool of people who vote is not fixed. We may know how many people are eligible to vote, but we can never know in advance the exact number who will actually cast ballots. It can happen — and often does — that a candidate will gain votes in a second electoral round without her opponent necessarily losing votes. This is because new voters appeared who did not participate in the first round. Campaign tactics can offer a pretty good idea of how candidates view an election. If they run negative political campaigns, that's a clue they consider elections to be a zero-sum game: They try to put down their opponent because they strongly associate the opponent's loss with their gain. What these candidates may not consider is that voters can choose not to vote.

The conventional approach to business is to portray it as a zero-sum game, in that any advantage gained by a rival represents a loss for your company. If following such an approach, companies would actively attempt to impede their competitors' successes. This zero-sum view produces monopolies, price and format wars, a lack of compatibility and interoperability and risks to sustainability. In reality, it's not necessarily true that "my rivals' profits are my losses." Even if you believe that global resources are limited, there are always opportunities to discover new ways of extracting resources more efficiently, finding innovative substitutes or coming up with new ideas about how to use or sell them.

Trade is not necessarily a zero-sum game. Consider one country that produces sunflower oil at a price of $2 a pound, while a second country produces sunflower oil of the same quality at $7 a pound. If the first country agrees to sell oil to the second at a price of $4 a pound, both countries benefit from the trade.

We live in a world in which uncertainties exist in many different forms. We know that a fire could burn our house down, that an accident could smash our car or that we could succumb to illness. Any one of these could inflict financial and personal damage, but instead of living with those fears, we buy insurance (home, car, life). If we're referring to these possibilities as zero-sum games, we misuse the term. Insurance transforms them into win-win situations: The insurance customer pays for protection, and the insurer makes a profit by investing the premiums.

BREAKING DOWN GAME THEORY

Games can be classified according to the number of strategies available to each player. If one player has m available strategies and another has n available strategies, then we will have an $m \times n$ game (or an $n \times m$ game, depending on the order of the players).

For simplicity, we will assume the game has two players: A and B.

The general approach is to present the payoffs of the game in a matrix like this one:

		Player A		
		I	II	III
Player B	1	L(I,1)	L(II,1)	L(III,1)
	2	L(I,2)	L(II,2)	L(III,2)

where:

- I, II, III, ... -> strategies available for player A
- 1, 2, ... -> strategies available for player B
- L(i, j) -> loss of player A (and gain to player B) when A chooses the strategy iє { I, II, III} and B picks the strategy jє{1, 2}

By convention, we will consider L(i, j) to have a positive sign if A is paying this amount to B and a negative sign if A is receiving this amount from B.

Here's an example of a zero-sum game:

Anne and Brandon are tossing two fair coins. If both are heads or both are tails, Anne gives $1 to Brandon. If one is heads and the other is tails, Anne receives $1 from Brandon.

The matrix of the payoffs will look like this:

		Anne	
		H	T
Brandon	H	1	-1
	T	-1	1

Because the expected net gain of each participant is 0, this is a fair game.

Strategy i is said to be dominant over strategy j if strategy i is at least as good as strategy *j.* If strategy i dominates strategy *j,* then we can eliminate strategy *j* because no rational player would choose it. For our situation with two players, player A will search for smaller losses (that is, smaller numbers) and player B will search for bigger payoffs (bigger numbers).

A player can take one of two approaches:

1. Search to maximize his minimum payoff..
2. Search to minimize his maximum loss.

In trading, the common approach is the second one, as it's a good way to control risk. If the worst happens, the player is prepared because she has estimated her maximum loss. This approach is the minimax criterion, after the von Neumann theorem. In other words, the minimax criterion minimizes the probability of high-impact risks.

The following matrix illustrates this approach:

		Player A		
		I	II	III
Player B	1	6	7	2
	2	3	2	-8
	3	8	4	3
	4	2	-5	1

We will take the role of player A.

If we choose strategy I, then the worst that can happen is losing 8.

If we choose strategy II, then the worst that can happen is losing 7.

If we choose strategy III, then the worst that can happen is losing 3.

Because we want to minimize the biggest loss that can happen, we will go with strategy III and assume a maximum loss of 3.

We say that strategies *i* and *j* are in equilibrium if the element L(i, j) is both the largest in its column and the smallest in its row. If such an element exists, we call it a saddle point. In the example above, the strategy (III, 3) is in equilibrium with 3 as the saddle point because it is the biggest element in column III and the smallest element in row 3.

A more general concept is the Nash equilibrium. This is a set of strategies in which every competitor's strategy is optimal, meaning the strategies of all other competitors are held constant. In other words, a Nash equilibrium occurs when no competitor can benefit by changing its strategy because all the other competitors' strategies remain unchanged. In the payoff matrix, the Nash equilibrium is one in which the row player's payoff is at least as large as all other payoffs in that column and the column player's payoff is at least as large as all other payoffs in that row. In the particular case of a zero-sum game, the Nash point would simultaneously be the largest in its column and the smallest in its row; therefore, it would be a saddle point.

Let's take the following example: Two TV channels (1 and 2) are competing for an audience of 100 viewers. The rule of the game

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is to simultaneously announce the type of show the channels will broadcast. Given the payoff matrix below, what type of show should channel 1 air?

		Channel 1		
		Movie	Reality Show	Stand-up Comedy
Channel 2	Movie	37	17	62
	Reality Show	47	60	52
	Stand-up Comedy	40	16	72

The payoff matrix says that if both channels decide to air a movie, then 37 viewers will watch the movie on 1 and 63 will watch on 2. Looking at the payoff matrix, the equilibrium point is 47 (movie on 1, reality show on 2) — the minimum value in its row and the maximum value in its column.

Not all payoff matrices have a saddle point. In this case, because the opponent's strategy is unknown, the optimal strategy will be a function of what the other player is doing. This is what is known as a randomized strategy.

To illustrate this situation, let's use a 2x2 payoff matrix:

		Player A	
		I	II
Player B	1	8	-2
	2	3	5

First, note that there is no saddle point and no dominant strategy. Let's say we bet on strategy I with probability p and on strategy II with probability *1-p*. If player B goes with option 1, our expected loss is 8p-2(1-p) = 10p-2. If player B is going with option 2, then our expected loss is 3p+5(1-p) = 5-2p. Our goal is to minimize our maximum loss, which occurs when 10p-2 = 5-2p, so p = 7/12. Therefore, we're betting on strategy I with probability 7/12 and on strategy II with probability 5/12. For a deeper dive into the statistics underlying zero-sum games, see the appendix.

CONCLUSION

Game theory and that subset known as zero-sum games are explorations into decision-making in a world that is uncertain, has limited resources and is less than transparent, particularly when it comes to what is going through the minds of other players. The notion of a "game" suggests there's a degree of competition taking place, often over those limited resources. The value of game theory is to understand a human situation — to quantify as far

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as possible the various outcomes of actions or strategies. Game theory is not about providing absolute answers but about gaining insights into how the actions of one player or multiple players affect each other. As von Neumann and Morgenstern realized, the zero-sum characteristic runs through a number of interactions, and it can be a valuable tool in understanding aggregations of human contestants, like markets.

One of the key assumptions of game theory is that players are rational. The assumption of rationality doesn't interfere with the possibility that a player may erroneously prefer some strategies over others. In determining their preferences and selecting their options, players can take into account available information, probabilities and potential costs and benefits. As in markets, players are rational if they have a clear view of their goals.

Rousseau used his story of the hunters to depict the dawning realization that in a state of nature a compact — what he called a social contract — could produce a mutual benefit. Like much of game theory, it's a deceptively simple story that grows more complex the more we think about it. Like many games, the outcome depends on the context. The hunters' decision to hang together represents an investment in a maximum payoff, and that's how it's usually portrayed. But it is just as possible that in a group that had recently dined, or was deeply distrustful of one another, or concerned about the danger of trying to kill a large deer possessing a formidable set of antlers, a risk-minimization strategy would win out. In short, each goal has different equilibria. Go for the rabbit today, and put off the stag hunt for another day. ■

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APPENDIX

In a statistical zero-sum game, one of the players is the nature — a player without a strategy, who acts arbitrarily — and the other is the statistician. The statistician has access to sample data to guide his decision, while the nature's choices are randomized. Consider the following game: The nature player picks a number *p*, which is either 0.3 or 0.6, and the statistician player has to guess that number. The rule of the game is that if the statistician wrongly concludes that *p* equals 0.3, he loses $1, and if he wrongly concludes that *p* equals 0.6, he loses $2; there is no reward if he is correct. His strategy is to observe a value *x* drawn randomly from the set S = {0,1,2} and to use one of the following decision functions:

$$d_1(x): \begin{cases} p = 0.3 \ \ if \ x = 0 \\ p = 0.6 \ if \ x = 1 \ or \ 2 \end{cases}$$

$$d_2(x): \begin{cases} p = 0.3 \ \ if \ x = 0 \ or \ 1 \\ p = 0.6 \ if \ x = 2 \end{cases}$$

$$d_3(x): \{ \quad p = 0.3 \ if \ x = 0{,}1 \ or \ 2$$

$$d_4(x): \{ \quad p = 0.6 \ if \ x = 0{,}1 \ or \ 2$$

where $d_k(x)$ is the output of his guess.

We call a risk function the function $R(d_k,j)=E[L(d_k(x),j)]$ where x is the observed value, d_k is a decision function and $L(d_k(x),j)$ is the loss suffered by the statistician if he makes the decision $d_k(x)$ and nature's strategy is *j*.

In our particular example, using the conditional probability, we have the maximum loss for each decision:

	d_1	d_2	d_3	d_4
p = 0.3	1.02	0.18	0	2
p = 0.6	0.16	0.64	1	0

The maximum loss the statistician can suffer is 1.02 if he uses the first decision function, 0.64 if he uses the second, 1 for the third and 2 for the last one. Because the statistician wants to minimize the maximum risk, he will use the second decision function.

As a general rule, the minimax criterion can be applied in statistical games by choosing the decision function d, for which the risk function R(d,j), maximized with respect to j, reaches the minimum over the set of decision functions.

ENDNOTES

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